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2006 J. Phys. A: Math. Gen. 39 8675

(http://iopscience.iop.org/0305-4470/39/27/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 8675–8680 doi:10.1088/0305-4470/39/27/008

Analogies between self-duality and stealth matter source

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Received 2 May 2006
Published 21 June 2006
Online at stacks.iop.org/JPhysA/39/8675

Abstract

We consider the problem of a self-interacting scalar field nonminimally coupled to the three-dimensional BTZ metric such that its energy–momentum tensor evaluated on the BTZ metric vanishes. We prove that this system is equivalent to a self-dual system composed by a set of two first-order equations. The self-dual point is achieved by fixing one of the coupling constant of the potential in terms of the nonminimal coupling parameter. At the self-dual point and up to some boundary terms, the matter action evaluated on the BTZ metric is bounded below and above. These two bounds are saturated simultaneously yielding to a vanishing action for configurations satisfying the set of self-dual first-order equations.

PACS number: 04.20.−q

1. Introduction

The self-duality is an important notion which has strong implications from a mathematical as well as physical point of view. In field theories, the notion of self-duality refers to models for which the set of second-order field equations can be reduced to a first-order one. This implies in general to select a particular form of the interactions, and also to relate the coupling constants of the problem between them. The resulting first-order equations are in general more simpler to analyse and correspond to the minimization of some functionals as the energy or the action.

The most fundamental self-dual model is the four-dimensional Euclidean Yang–Mills theory for which the action is minimized by the first-order (anti) self-dual equations

$$F_{\mu\nu} = \pm\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho},$$

and whose solutions are known as the instantons. In Euclidean signature, the Yang–Mills energy–momentum tensor vanishes identically for fields satisfying the self-duality equations.

Another important example of self-dual theory is given by the three-dimensional Abelian–Higgs model which has a minimum energy self-dual solutions which correspond to vortices. In this case, the Bogomolnyi self-dual point is defined by $\lambda \propto e^2$, where λ is the coupling constant of the symmetry breaking quartic potential and e is the electric charge of the scalar field. The functional energy evaluated at this point is bounded and the bound is saturated for fields satisfying the first-order self-dual equations [1, 2]. However, the first-order equations are not solvable and no explicit solutions are known even in the radial symmetric case. In three dimensions, (non)relativistic scalar fields minimally coupled to a gauge field whose dynamics is governed by a Chern–Simons Lagrangian also exhibit a self-dual behaviour [3–5]. These self-dual Chern–Simons theories possess vortex solutions and are relevant in the study of planar phenomena as the quantum Hall effect or the high temperature superconductivity. In the (non)relativistic cases, the self-duality is achieved provided the introduction of a particular form of the potential with a strength potential fixed in terms of the Chern–Simons coupling constant and the electric charge of the scalar field. In both situations, at the self-dual point the energy is bounded below and the bound is saturated by the first-order self-dual configurations. In contrast with the Yang–Mills model, the energy–momentum tensor of these self-dual scalar field theories, namely the Abelian–Higgs model and the Chern–Simons theories, does not vanish for the self-dual configurations; for detailed reviews on Chern–Simons self-duality see [6–8]. There also exists self-dual models obtained by considering at the same time the Maxwell term and the Chern–Simons term, see e.g. [9–11].

In this paper, we shall be concerned by the problem of a self-interacting scalar field nonminimally coupled to the three-dimensional BTZ metric [12] such that its energy–momentum tensor $T_{\mu\nu}$ evaluated on the BTZ metric vanishes. In this case, the set of equations is given by the conditions $T_{\mu\nu} = 0$ evaluated on the BTZ metric while the scalar field satisfies a nonlinear Klein–Gordon equation. This problem has already been solved by a direct integration of the second-order field equations [13]. In this reference, it has been proved that these configurations, the so-called *stealth* configurations, exist only for a zero angular momentum and for a particular form of the potential. This latter is a local function of the scalar field and is given by the sum of three different exponents of the scalar field with two arbitrary coupling constants while the remaining one is fixed in terms of the nonminimal coupling parameter.

The purpose of this paper is to establish a certain analogy between this particular problem of stealth matter and those related to self-duality models. In particular, we shall prove that the stealth matter configuration is equivalent to a self-dual system given by a set of two first-order equations. In the analogy with self-dual model, the Bogomolnyi self-dual being achieved by fixing one coupling of the potential in terms of the nonminimal coupling parameter. Moreover, we will show that, up to some boundary terms, the matter action evaluated on the BTZ background is bounded below and above and both bounds are saturated simultaneously at the self-dual point for configurations satisfying this set of self-dual first-order equations.

2. Stealth scalar field over the BTZ black hole

The fundamental tenet of general relativity is the manifestation of the curvature of spacetime produced by the presence of matter. This phenomena is encoded through the Einstein equations that relate the Einstein tensor (with or without a cosmological constant) to the energy–momentum tensor of the matter,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \qquad (2a)$$

Since the energy–momentum tensor depends explicitly on the metric, both sides of the equations must be solved simultaneously. However, one can ask the following question: for a fixed geometry solving the vacuum Einstein equations, is it possible to find a matter source coupled to this spacetime without affecting it? Concretely, this problem consists of examining a particular solution of the Einstein equations (2*a*) for which both sides of the equations vanish, i.e.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 = \kappa T_{\mu\nu}. \tag{2b}$$

In three dimensions, such gravitationally undetectable solutions have been obtained in the context of scalar fields nonminimally coupled to gravity with a negative cosmological constant [14]. The same problem has also been considered in higher dimensions but with a flat geometry [15]. More recently, it has been considered the coupling of gravity to a matter field action for which the Lagrangian density is a power of the massless Klein–Gordon Lagrangian. In this case, the existence of nontrivial scalar field configurations with vanishing energy–momentum tensor on any static, spherically symmetric vacuum solutions of the Einstein equations has been shown [16].

Here, we are concerned with a nontrivial example of stealth matter which consists of a three-dimensional self-interacting scalar field nonminimally coupled to the BTZ black hole metric. The corresponding action is given by

$$\begin{aligned} S &= \int \mathrm{d}^3x\, \sqrt{-g}\left(\frac{1}{2\kappa}(R + 2l^{-2}) - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{2}\xi R\Phi^2 - U(\Phi)\right), \\ &= S_{\mathrm{EH}} - S_M, \end{aligned} \tag{2c}$$

where $\Lambda = -l^{-2}$ is the cosmological constant, ξ is the nonminimal coupling parameter and $U(\Phi)$ is the self-interaction potential that reads

$$U(\Phi) = \lambda_1\Phi^2 + \lambda_2\Phi^{(1-2\xi)/\xi} + \lambda_3\Phi^{1/(2\xi)}, \tag{2d}$$

where λ_1, λ_2 and λ_3 are the arbitrary constants. The action S_{EH} refers to the Einstein–Hilbert action with cosmological constant while the matter action S_M is given by

$$S_M(g,\Phi) = \int \mathrm{d}^3x\, \sqrt{-g}\left(\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi + \frac{1}{2}\xi R\Phi^2 + U(\Phi)\right). \tag{2e}$$

The field equations obtained by varying the metric in action (2*c*) are

$$G_{\mu\nu} - l^{-2}g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2f}$$

while the variation of the scalar field yields a generalized Klein–Gordon equation

$$\Box\Phi = \xi R\Phi + \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}, \tag{2g}$$

where the energy–momentum tensor is given by

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - g_{\mu\nu}\left(\tfrac{1}{2}\nabla_\alpha\Phi\nabla^\alpha\Phi + U(\Phi)\right) + \xi(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\Phi^2. \tag{2h}$$

We posit the metric to be the BTZ metric without angular momentum[1],

$$\mathrm{d}s^2 = -\left(\frac{r^2}{l^2} - M\right)\mathrm{d}t^2 + \left(\frac{r^2}{l^2} - M\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \tag{2i}$$

where M is the mass of the black hole. Since this metric is a solution of the (2 + 1) vacuum Einstein equations with a negative cosmological constant, our set of equations (2*f*)–(2*g*) reduce to the finding of a scalar field Φ such that the energy–momentum tensor (2*h*) evaluated on the BTZ metric vanishes, i.e. (2*b*).

[1] In [13], it has been shown that the existence of stealth matter field requires the absence of the angular momentum.

We now posit a set of two first-order equations,

$$\partial_t \Phi = -\frac{2K\xi}{1-4\xi}\sqrt{MF(r)}\sqrt{\frac{(\Phi^{(4\xi-1)/(2\xi)}-h)^2}{K^2l^2F(r)}-1}\,\Phi^{(1-2\xi)/(2\xi)}, \qquad (2j(a))$$

$$\partial_r \Phi = -\frac{2\xi}{1-4\xi}\Gamma^t_{tr}(\Phi - h\Phi^{(1-2\xi)/(2\xi)}), \qquad (2j(b))$$

where $F(r)$ is the BTZ metric function, $F(r) = r^2/l^2 - M$ and K and h are two constants related to the coupling constant λ_2 as

$$\lambda_2 = \frac{2\xi^2}{l^2(1-4\xi)^2}(h^2 + K^2l^2M).$$

After a tedious but straightforward computation, one obtains that for a scalar field satisfying the set of first-order equations (2$j(a)$), the energy–momentum tensor evaluated on the BTZ metric becomes

$$T_{\mu\nu} = g_{\mu\nu}\left[\Phi^2\left(\frac{2\xi^2}{l^2(1-4\xi)} + \lambda_1(4\xi-1) + \frac{\xi}{l^2}(1-12\xi)\right)\right], \qquad (2k)$$

where in order to derive this expression, we have used that the Christoffel symbols associated with the BTZ metric satisfy

$$\left(\Gamma^t_{tr}\right)^2 = \frac{g_{rr}}{l^2}\left(1+\frac{M}{F(r)}\right), \qquad \partial_r\Gamma^t_{tr} = -\frac{g_{rr}}{l^2}\left(1+\frac{2M}{F(r)}\right),$$

$$\Gamma^t_{tr}\Gamma^r_{rr} = -\frac{g_{rr}}{l^2}\left(1+\frac{M}{F(r)}\right), \qquad \Gamma^t_{tr}\Gamma^r_{tt} = -\frac{g_{tt}}{l^2}\left(1+\frac{M}{F(r)}\right), \qquad \Gamma^t_{tr}\Gamma^r_{\theta\theta} = -\frac{g_{\theta\theta}}{l^2}.$$

Hence, for scalar fields satisfying the set of first-order equations (2$j(a)$), the condition $T_{\mu\nu} = 0$ is verified provided the coupling constant λ_1 to be fixed in terms of the nonminimal coupling parameter as

$$\lambda_1 = \frac{\xi}{l^2(1-4\xi)^2}(1-8\xi)(1-6\xi), \qquad (2l)$$

while the third coupling constant λ_3 is fixed by solving the generalized Klein–Gordon equation. The resulting potential becomes exactly the one derived in [13],

$$U(\Phi) = \frac{\xi}{l^2(1-4\xi)^2}\big((1-8\xi)(1-6\xi)\Phi^2 + 2\xi(h^2+K^2l^2M)\Phi^{(1-2\xi)/\xi} + 4\xi(1-8\xi)h\Phi^{1/(2\xi)}\big). \qquad (2m)$$

Note that relation (2l) is analogue to the Bogomolnyi self-dual point where the coupling constants of the problem are in general related between them.

In sum, we have shown that the scalar configuration given by a scalar field satisfying the self-dual equations (2$j(a)$) together with the potential (2m) is also solution of the stealth equations evaluated on the BTZ metric (2f)–(2g). It is easy to solve the self-dual equations, and the solution is given by

$$\Phi(t,r) = \left(\cosh(\sqrt{M}t/l)\sqrt{r^2 - Ml^2} + h\right)^{2\xi/(4\xi-1)} \qquad (2n)$$

which agrees with the solution obtained in [13].

As said in the introduction, the self-dual equations in self-dual models correspond in general to the minimization of some functionals of the problem as the energy or the action. In our case, we prove that the matter action evaluated on the BTZ metric is bounded below and above up to some surface terms and these two bounds are achieved in the self-dual point (2l) for configurations satisfying the self-dual equations (2j(a)).

The matter action (2e) evaluated on the BTZ metric at the self-dual point (2l) reads

$$S_M = \int r\left(-\frac{1}{2F(r)}(\partial_t\Phi)^2 + \frac{1}{2}F(r)(\partial_r\Phi)^2 - \frac{3\xi}{l^2}\Phi^2 + U(\Phi)\right)\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}\theta, \tag{2o}$$

where U refers to the self-dual potential (2m) and the scalar curvature is given by $R = -6/l^2$. It is interesting to note that the functional (2o) can be rewritten as

$$\begin{aligned}\tilde{S}_M = 2\pi\int \mathrm{d}r\,\mathrm{d}t\Bigg\{&\frac{1}{2}rF(r)\left[\partial_r\Phi + \frac{2\xi}{1-4\xi}\Gamma^t_{tr}(\Phi - h\Phi^{(1-2\xi)/(2\xi)})\right]^2\\ &-\frac{r}{2F(r)}\left[\partial_t\Phi + \frac{2K\xi}{1-4\xi}\sqrt{MF(r)}\sqrt{\frac{(\Phi^{(4\xi-1)/(2\xi)}-h)^2}{K^2l^2F(r)} - 1}\,\Phi^{(1-2\xi)/(2\xi)}\right]^2\Bigg\},\end{aligned} \tag{2p}$$

where in order to derive this expression we have first operated some integrations by parts and then dropping all the surface terms. From expression (2p), it is easy to see that the functional $\tilde{S}_M$ is bounded below and above and both bounds are simultaneously saturated yielding to a vanishing action, i.e. $\tilde{S}_M = 0$, for configurations satisfying the set of self-dual equations (2j(a)).

3. Conclusions

Here, we have addressed the problem of the analogy that may exist between self-duality and a stealth matter configuration. We have considered the problem of a self-interacting scalar field nonminimally coupled to the BTZ metric. The resulting stealth matter field equations reduce to the vanishing of the energy–momentum tensor expressed on the BTZ metric together with a generalized Klein–Gordon equation. The potential is given by the sum of three different exponents of the scalar field with two arbitrary coupling constants while the remaining one is fixed in terms of the nonminimal coupling parameter. We have exhibited some analogy between this problem and those arising in the context of self-dual theories. In particular, the coupling constant of the potential fixed in terms of the nonminimal coupling parameter is shown to play the role of the Bogomolnyi self-dual point. In addition, we have exhibited a set of first-order equations and prove that any scalar field satisfying these self-dual equations at the self-dual point is also solution of the stealth equations. Moreover, we have shown that at the self-dual point, the matter action evaluated on the BTZ metric is bounded below and above, up to some surface terms that we have dropped. Curiously enough, these two bounds are saturated simultaneously yielding to a vanishing action for configurations satisfying the set of self-dual first-order equations.

In the self-dual models, the problem of the existence and uniqueness of the self-dual solutions is a nontrivial issue. For example, it is well known that in the Abelian Higgs model, all the finite energy solutions of the full second-order field equations are also solutions to the first-order self-duality equations. In our case, due to the uniqueness of the solution of the starting problem [13], one can also conclude to the uniqueness of the self-dual solution.

In order to have a more deep connection between the self-duality and the stealth configuration, it would be desirable to explore the issue of the supersymmetric extension. Indeed, in the self-dual models, the self-dual point is also the point at which the theory can be extended to a supersymmetric model. An interesting work will consist to see whether the self-interacting scalar field nonminimally coupled to the BTZ metric can be extended to a supersymmetric theory.

Acknowledgments

We thank an anonymous referee for bringing our attention on this problem. We are also grateful to the staff of the CECS for useful discussions. This work is partially supported by grants 1051084, 1060831 from FONDECYT. Institutional support to the Centro de Estudios Científicos (CECS) from Empresas CMPC is gratefully acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

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